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ON MODEL IDENTIFICATION PROBLEMS IN ROCK MECHANICS

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SUMMARY – The analysis of a structural system implies the generation of a mathematical model of it. Inaccuracies of analysis are often due primarly to insufficient information on material behaviour. Experimental data concerning the overall response of the actual system can be used to identify material characteristics incorporated in the mathematical model. The identification problems considered in this paper concern elastic moduli, yield surfaces, and, marginally, some aspects of external actions, with reference to certain situations in geotechnical engineering (rock excavations and embankments). Three solution method are formulated and discussed: an iterative least square procedure, a simplex search strategy, a mathematical programming approach.

1. INTRODUCTION

After the recent developments in computer technology and numerical methods, most linear and many nonlinear boundary value problems which arise in engineering, can be solved with practically sufficient approximation, e. g. on the basis of finite element or finite difference discretizations.

The main discrepancies between theoretically predicted and experimentally observed responses are frequently due to unrealistic assumptions concerning the local deformability ("constitutive") laws, rather than to inaccuracies in the numerical solutions of the mathematical problems formulated in the context of continuum mechanics. This occurs particularly in geotechnical engineering and, more specifically, in rock mechanics. In fact, rock masses exhibit, as a rule, an highly nonhomogeneous and anisotropic nature (due also to macrofractures, faults, and relevant filling materials (Jaeger, 1972)); moreover, their local geometric and physical properties are scarcely known in real-life situations. Therefore, experimental results obtained on the basis of normal laboratory specimens are hardly meaningful or useful in order to predict overall responses of such geotechnical systems to loading processes.

As a contribution to overcome this difficulty, in this paper some typical engineering problems of stress analysis in rock structures are tackled and discussed by a "system identification" point of view.

The meaning of identification to the present purposes is illustrated by fig. 1, in schematic, general terms. The mathematical model adopted for theoretical predictions on the actual system behaviour contains some available parameters which characterize physical properties and are not susceptible of direct measurements.

These additional unknowns (additional with respect to "normal" analysis problems) are determined on the basis of a comparison between predicted values and experimental values concerning the overall responses of the model and the overall response of the actual system, respectively, to the same, suitably chosen, external actions ("input"). The above comparison implies to define

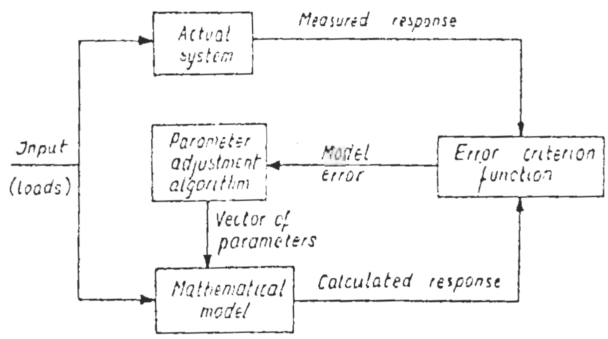

Fig.1 – An illustrative scheme of parametric identification problems.

a measure of the discrepancy or "error criterion function" and to choose the parameters which minimize this discrepancy. Subsequently the model thus "identified" is apt to predict "a priori" (i.e. without any more experimental information to feed in) the overall response to other external actions. Clearly, the reliability of these predictions depends not only on the accuracy of the solution to the identification problem; it depends also on three important factors: (a) the realism of the mathematical model adopted (i.e. of its qualitative and quantitative aspects which were assumed, not obtained by identification); (b) the closeness between the inputs considered in the identification process and in the subsequent use of the model (i.e. it is hard to obtain and even to conceive a simulation accurate over a very broad range of arbitrarily different inputs and stress and strain fields), (c) the accuracy of the measurements.

Identification problems have been extensively studied in system science and control theory (see e.g. Distefano, 1974, Eykhoff, 1974), in a more general context than that outilined above (e.g. time-dependent dynamical systems; systems with functions to identify instead of parameters; stochastic inputs, etc.). In structural and applied mechanics, identification concepts in the above sense were previously used, to the author's knowledge , in a limited set of cases, among which the following ones seem particularly worth mention: (a) "calibration" of the linear elastic and thermoelastic mathematical model of a dam-bedrock system for continuing (in "real-time") control of in-service behaviour of dams (Bonaldi et al., 1975); (b) in biomechanics, determination of phenomenological models for the time-dependent rheological behaviour of living tissues (Pister-Distefano, 1970); (c) evaluation of damping characteristics of frames for dynamic analysis under seismic loading ,(Ibañez - 1973); (d) characterization of rubberlike materials for large strains elastic analysis (Iding et al., 1974). It is worth noting that also "parametric analysis" (for several values of a poorly known parameter in order to single out the most appropriate one) is in principle a special, trivial case of identification.

The distinctive features of the identification problems in geotechnical engineering, considered in this paper, can be pointed out as follows: (i) the number of (time-independent) material parameters to identify is small, say ranging from two to ten, but the number of accompanying variables (such as nodal displacements of a finite element model) is large; (ii) for given parameters, the analysis of the system can be performed numerically, not analytically, by means of the model in a relatively economical and efficient way; (iii) experimental information which can be gathered concerns usually relative displacements in some nodes for one or a few loading conditions: (iv) some parameters to identify may be contained also in the loading condition (i.e. in the "input" of fig. 1), since "in situ", tectonic stresses are sometimes the main sources of static boundary conditions(e.g.in excavation analysis) and often cannot be measured directly with adequate accuracy.

The purpose pursued in the present study, preliminary to a broader research including numerical tests and applications, is to formulate and to discuss comparatively some identification procedures for problems with the above peculiarities, under two "qualitative" basic assumptions on the (time-independent) behaviour of the material or materials: (I) linear elasticity (isotropic or anisotropic); (II) perfect or "ideal" elastoplasticity. Either material idealization proves satisfactory and is customary in engineering analysis of rock masses and of several categories of earth structures.

2. TYPICAL ENGINEERING SITUATIONS

Two geotechnical engineering situations where the above identification problems may arise, are briefly described below in order to fix ideas.

(A) The excavation of a large cavern, e.g. for an underground power plant, proceeds by layers (fig. 2).

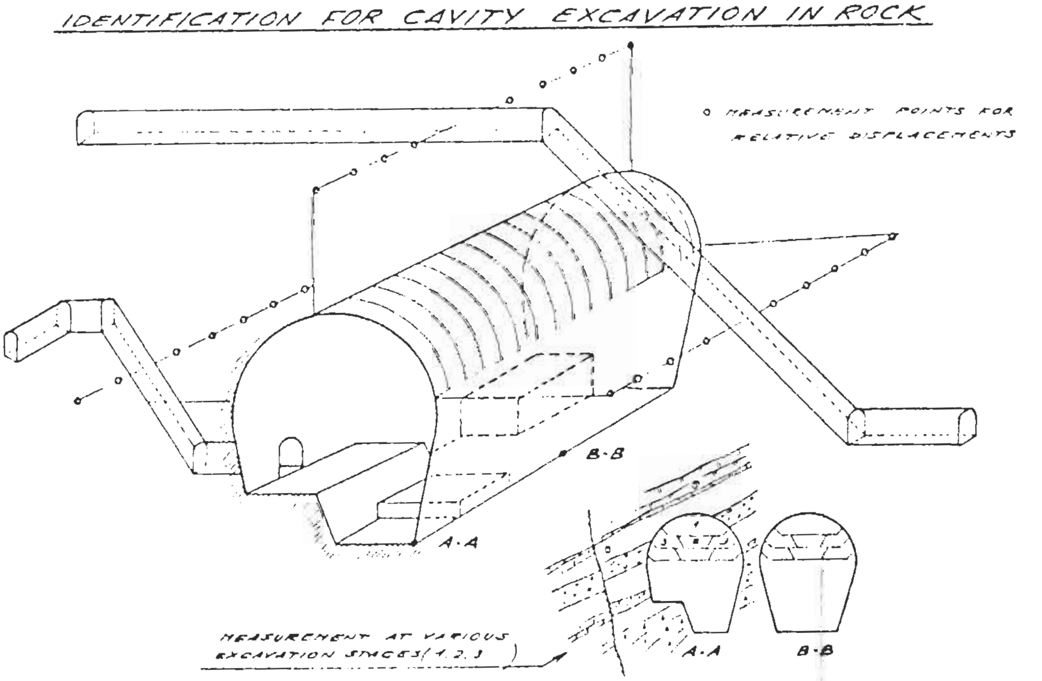

Fig. 2 - A cavern in rock (schematically); layer excavation sequence and location of (relative) displacement measurements for model identification.

Relative displacements generated by the earlier excavation stages can be, and usually are, measured by means of instruments placed in small neighbouring, previously worked out tunnels (Abraham-Pahl, 1976). The identification process performed on this basis is of use for more accurate calculations of the overall response to the later excavation stages. The loading conditions in early stages are given mainly by the preexisting, residual stresses carried by the material removed. Parameters which define the "in situ" stress distribution (two parameters according to the classical Heim-Rzika hypothesis) are hard to measure directly; hence, some of them might be involved as unknows in the identification process, as substitution or check of direct measurements by means of "ad hoc" gauges in drill holes. A situation similar to the preceding one is tunnelling in rock (fig. 3) where identification performed on the basis of relative displacement measurements in transversal drill holes during one stage of excavation front advancement can be useful for calculating deformations due to the subsequent stage.

(B) Embankments by successive layers or "lifts", such as the construction of gravity rockfill or earth dam, (fig. 4-see next page) gives rise to identification in the sense specified in the Introduction. In fact, the difficulty in the statistical analysis and design of a major embankment dam, rests primarly on the use of parameters describing the stress-strain behaviour of the cohesionless fill material, particularly when large rock fragments are employed (besides the central clay core or similar impervious parts). The linear-elastic idealization seems sufficient to predict deformations that occur during the construction and with the reservoir filling, but laboratory testing of the rockfill materials turn out to be inadequate even with sophisticated and costly experimental methods (Charles, 1976). Therefore, identification of (apparent) elastic moduli through "in situ" measurements as shown in fig. 4 might be a proposition of practical use.

3. GOVERNING RELATIONS

In order to provide a clearer insight into the mechanical and mathematical nature of the identification problems in point, the simplest finite element discretization will be referred to in what follows.

Let element i (i = 1...m) be a constant-strain, homogeneous , four-nodes tetrahedron (three-nodes triangle in plane problems (Zienkiewicz, 1971)). The element deformations will be governed by the 6 component vector (3 component in the plane) $\underline{q}_i$ of "natural" or intrinsic generalized strains, i.e. of edge (respectively, side) elongations, unaffected by rigid body motions (*). Correspondingly, natural generalized stresses $\underline{Q}_i$, in terms of selfequilibrated pairs of forces along each edge (side), are adopted; thus, the dot product $\underline{Q}_i^T\ \delta\underline{q}_i$ represents the first-order virtual work performed over the element volume V_i. With these assumptions, the generalized (element) strains and stresses are related to the actual (material) strains $\underline{\varepsilon}_i$ and stresses $\underline{\sigma}_i$, respectively, through linear, nonsingular, contragradient transformations:

$$\underline{\varepsilon}_i = \underline{T}_i\ \underline{q}_i \qquad V_i\ \underline{T}_i^T\ \underline{\sigma}_i = \underline{Q}_i \tag{1}$$

where $\underline{T}_i$ is a matrix which depends on the element geometry alone (6 x 6 in threedimensional, 3 x 3 in twodimensional cases).

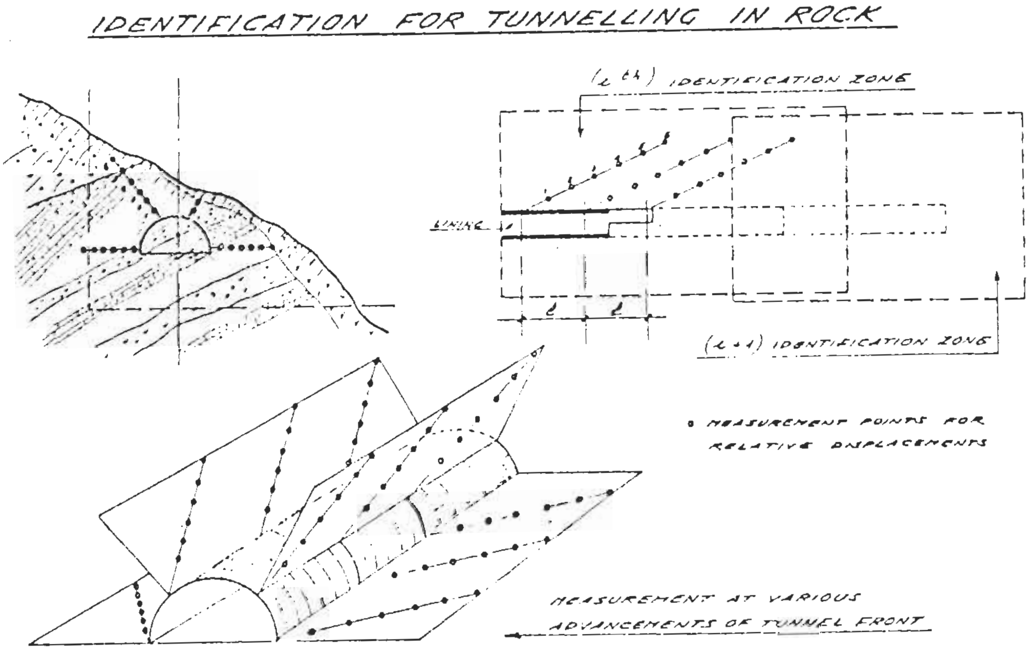

Fig. 3 - Tunnelling in rock (schematically): stages of excavation front advancement and arrangement of gauge sets.

(*) Notation: matrices (and vectors) are denoted henceforth by underlined symbols;a superscript T means transpose,0 is a matrix (or vector)whose entries are all zero;vector inequalities apply to each pair of corresponding components.

Let $\underline{q}^T = (\underline{q}_1^T \ldots \underline{q}_m^T)$ and $\underline{Q}^T = (\underline{Q}_1^T \ldots \underline{Q}_m^T)$ denote vectors which include as subvectors the above $\underline{q}_i$ and $\underline{Q}_i$ for all m finite elements of the model. The d-vectors $\underline{u}$ will be formed with all d displacements at free (not constrained) nodal points in the mesh, i.e. with all degrees of freedom of the assembled discrete model.

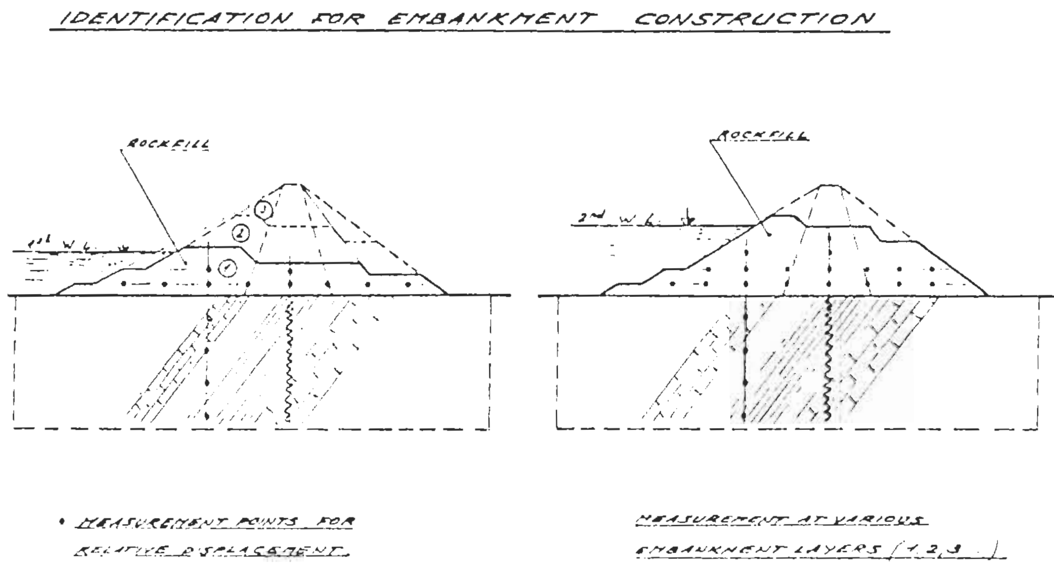

Fig. 4 - Rockfill-earth dam (schematically): relative displacements measured at some construction stages with instruments placed in the filling and in the bedrock.

Correspondingly, $\underline{F}$ denotes the d-vector of nodal forces equivalent (in the virtual work, sense) to body forces over the system and surface forces on its boundary. With thse symbols the geometric compatibility and the equilibrium equations read

$$\underline{q} = \underline{C}\,\underline{u} \qquad (2)$$

$$\underline{C}^T\ \underline{Q} = \underline{F} \qquad (3)$$

where the compatibility matrix $\underline{C}$ (6 m X d) depends only on geometric data of the finite element mesh and its fixity constraints. Rigid body motions are ruled out, so that the rank of $\underline{C}$ is d.

The first basic hypothesis (I) on material behaviour is the linear elastic idealization:

$$\underline{\sigma} = \underline{E}\,\underline{\varepsilon} \qquad (4)$$

In Eq. (4) $\underline{E}$ denotes the symmetric, positive definite matrix of elastic moduli. Its entries can be expressed as linear functions of e independent quantities: e = 2 under the isotropy assumption; e = 5 for rotational orthotropy, i.e. for stratified media, isotropic in the (known) plane of the strata; e = 9 for orthotropy with three known orthogonal symmetry planes; e = 21 for general anisotropy (Allirot, 1976).

Some or all these quantities are included in the vector $\underline{P}$ of the n parameters to identify. For the sake of simplicity, the model simulating the system will be assumed as homogeneous ($\underline{E}$ equal in all elements), without loss of generality since the extension to models with parts of different materials to identify is straightforward.

The second, alternate hypothesis (II) on material behaviour is the elastic, perfectly plastic idealization. The shape of the yield surface in the stress space will be regarded as the main object of the identification process. To this purpose (like to other purposes, (Maier, 1976)), it appears convenient to adopt a piecewise linear (polyhedrical) approximation of the yield locus, i.e. to define it by linear inequalities:

$$\underline{\varphi} \equiv \underline{n}^T\,\underline{\sigma} - \underline{k} \leq \underline{0} \qquad (5)$$

where, (see fig. 5): $\underline{\varphi}$ denotes a y-vector of yield functions; $\underline{n}$ is a matrix, each column of which (say the r-th) represents the outward unit vector $\underline{n}_r$ normal to the r-th yield plane (r = 1...y); $\underline{k}$ is the vector of the distances of these planes from the origin. These distances are parameters P_r to identify. Of course, symmetry requirements will usually reduce the number of independent P_r. Moreover only the yield planes which are likely to be active need to be considered.

If the rule of normality to the yield surface is assumed for plastic strain vectors (associative flow laws, material stability in Drucker's sense), the strain vector can be expressed as:

$$\underline{\varepsilon} = \underline{E}^{-1}\underline{\sigma} + \underline{n}\,\underline{\lambda} \qquad \underline{\lambda} \geq \underline{0} \qquad (6)$$

In Eq.(6) elastic and plastic strains are separated, the latter being governed by the "plastic multipliers" $\underline{\lambda}^T \equiv (\lambda_1 \ldots \lambda_y)$.

Local unloadings usually do not affect significantly the overall response of an elastoplastic system to proportional external loading processes. Therefore, with the restriction here assumed to such process, the irreversibility of plastic yielding can be ignored and holonomic (in finite terms) constitutive laws can be assumed, in the spirit of the deformation theory of plasticity. Laws of this type are formulated simply by

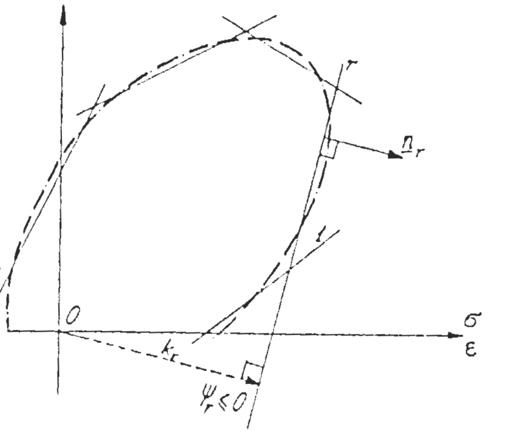

Fig. 5 - Piecewiselinear approximation of yield surfaces in the stress space.

supplementing Eqs. (5) (6) with the non linear equation

$$\underline{r}^T \underline{\lambda} = 0 \qquad (7)$$

Since Eq. (7) concerns vectors definite in sign, it implies that in each pair of corresponding components at least one vanishes, i.e. it represents a complementary condition (not only an orthogonality requirement).

4. ELASTIC MODELS IDENTIFIED BY CONDENSATION AND LEAST-SQUARE ITERATIVE PROCEDURE

The method presented below is a modified version of that proposed in (Kavanagh, 1973), adjusted to the requirements of the present category of problems. The elastic stiffness matrix $\underline{E}_i'$ of the i-th finite element is readily obtained from Eqs. (1) (4); diag$[\underline{E}_i']$ being the block-diagonal matrix of these matrices, the equilibrium equation of the model can be expressed through Eqs. (2) (3) in the form:

$$\underline{F} = \underline{\bar{K}}\,\underline{u} \qquad (8)$$

having set:

$$\underline{\bar{K}} \equiv \underline{C}^T \,\text{diag}\,[\underline{E}_i']\,\underline{C} = \sum_{1}^{n}{}_h\, P_h\, \underline{\bar{K}}_h \qquad (9)$$

P_h being the h-th material elastic constant to identify ($h = 1 \ldots n$). The latter of Eqs. (9) makes explicit the following circumstance, which clearly follows from the former and from the linear dependence of the material elastic tensor $\underline{E}$ on P_h : the (assembled, positive definite) elastic stiffness matrix of the model can be expressed as a linear combination, through coefficients P_h, of matrices $\underline{\bar{K}}_h$; the h-th of these represents the stiffness of the model with a fictitious material characterized by $P_h = 1$, $P_s = 0$ for $s \neq h$.

Differences between displacements at some nodes will be assumed as quantities to measure in the actual system. Let these measurable, relative displacements be collected in a vector $\underline{v}$ and the nonmeasurable (absolute) displacements in vector $\underline{w}$. We may write:

$$\left\{\begin{array}{c}\underline{v}\\ \hline \underline{w}\end{array}\right\} = \underline{G}\,\underline{u} \qquad (10)$$

It is always possible to choose the $\underline{v}$ and $\underline{w}$ variables in such a way that they are independent and they uniquely define the model deformed configuration $\underline{u}$, so that det$\underline{G} \neq 0$.

Thus Eqs. (8) (9) can be re-written as:

$$\underline{F} = \sum_{1}^{n} P_h\,\underline{K}_h \left\{\begin{array}{c}\underline{v}\\ \hline \underline{w}\end{array}\right\}, \text{ where } \underline{K}_h \equiv \underline{\bar{K}}_h\,\underline{G}^{-1} \qquad (11)$$

and subsequently can be partitioned in the form:

$$\left\{\begin{array}{c}\underline{V}\\ \hline \underline{W}\end{array}\right\} = \sum_{1}^{n}{}_h\, P_h \left[\begin{array}{c|c}\underline{K}_h^{vv} & \underline{K}_h^{vw}\\ \hline \underline{K}_h^{wv} & \underline{K}_h^{ww}\end{array}\right] \left\{\begin{array}{c}\underline{v}\\ \hline \underline{w}\end{array}\right\} \qquad (12)$$

where $\underline{V}$ and $\underline{W}$ are subvectors of the given load vector $\underline{F}$ partitioned in accordance to $\underline{v}$, $\underline{w}$. Note that $\underline{K}$ is not symmetric in general, through still nonsingular. Let a static condensation be performed on Eq. (12), eliminating the non measurable displacements $\underline{w}$. Simple algebraic manipulations lead to:

$$\underline{V} - \underline{R}\,\underline{W} = \left(\sum_{1}^{n}{}_h\, P_h\,\underline{K}_h^{vv} - \underline{R}\sum_{1}^{n}{}_h\, P_h\,\underline{K}_h^{wv}\right)\underline{v} \qquad (13)$$

in which:

$$\underline{R} \equiv \sum_{1}^{n}{}_h\, P_h\,\underline{K}_h^{vw}\left(\Sigma_h\, P_h\,\underline{K}_h^{ww}\right)^{-1} \qquad (14)$$

The matrix in brackets in Eq. (14) is nonsingular (although each of its addends may be singular), as long as the parameter vector $\underline{P}$ defines an elastic material (with positive definite elastic tensor). In fact, the quadratic form associated to it represents the strain energy stored in the model acted upon through the d degrees of freedom w, while the d_v measurable relative displacements are kept constant by additional constraints.

If experimental values $\underline{v}^*$ are fed in Eq. (13), this combined with (14) becomes a set of d_v nonlinear equations in the n unknows P_h. Clearly, the number d_v of independent information on the actual system must be not less than the number n of available parameters in its mathematical model. Eqs. (13) are linear in $\underline{P}$ if matrix $\underline{R}$ is calculated on the basis of an assumed vector $\underline{P}$. This suggest to adopt an iterative procedure combined with the least-square method to find the best-fitting solution to the overdetermined ($d_v > n$) equation system (14). Setting:

$$\underline{H}_h \equiv \underline{K}_h^{vv} - \underline{R}\;\underline{K}_h^{wv} \quad (h = 1 \ldots n) \qquad (15)$$

$$\underline{L} \equiv [\underline{H}_1\,\underline{v}^* \,|\, \ldots \,|\, \underline{H}_n\,\underline{v}^*] \qquad (16)$$

Eq. (13) acquires the form:

$$\underline{V} - \underline{R}\,(\underline{P})\,\underline{W} = \underline{L}\,(\underline{P})\,\underline{P} \qquad (17)$$

The least-square solution to (17) for given $\underline{R}$ and $\underline{L}$ provides quite naturally the iteration scheme to overcome the nonlinearity (Kavanagh K., 1973):

$$P^{j+1} = [L^T(P^j)\, L(P^j)]^{-1}\, L^T(P^j)[\,V - R(P^j)\, W] \qquad (18)$$

In Eq. (18), superscript j is the step index in the iteration sequence, which is initialized by a reasonably assumed vector P^1.

The method is outlined in the flow chart of fig. 6.

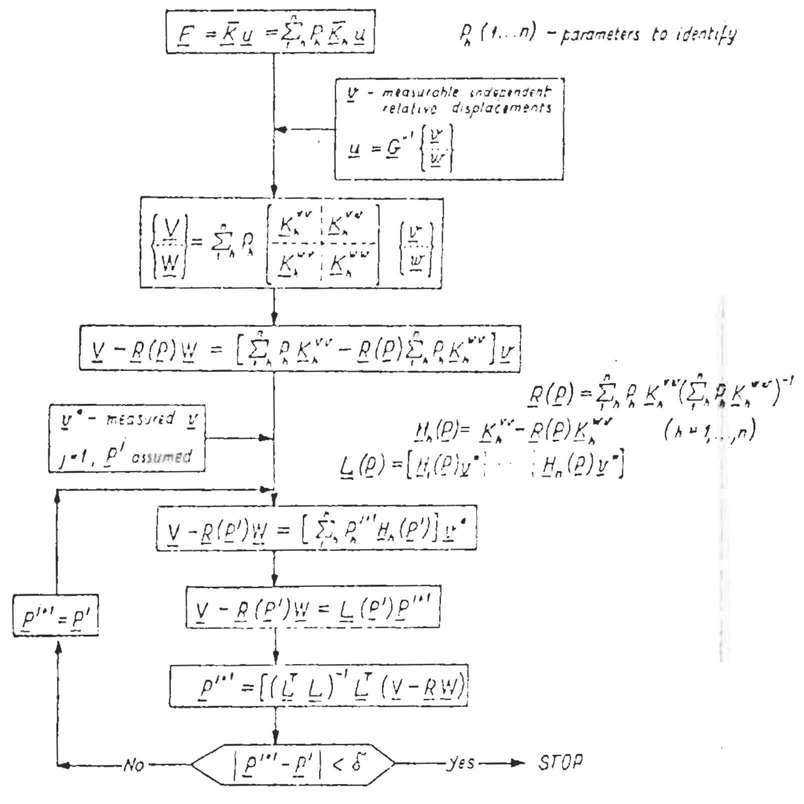

Fig. 6 - Outline of the least-square, iterative method for elastic model identification; δ = suitably chosen tolerance.

Its convergence and speed of convergence has to be assessed numerically, but some extensive tests presented in (Kavanagh K., 1973) on similarly structured problems appear encouraging.

5. MODEL IDENTIFICATION BY A "SIMPLEX" STRATEGY

The same problem as in Sec. 4 is tackled here by a different approach, resting on the following remarks. The value of a discrepancy measure or criterion function ϕ can be efficiently calculated (through a linear finite element analysis) for any choice of the available parameters P_h. The numerically defined function ϕ (P) turns out to be highly nonlinear and such that its derivatives are difficult to evaluate. However, in well posed problems it can be conjectured that it has a single minimum at least within the domain of admissible P_h (i.e. in the range imposed by physical consistency, such as e.g. $E > 0$, $0 \leq \nu \leq 0{,}5$ for isotropic materials). Therefore, the minimization can be performed as a sequence of normal analyses and comparisons; organized according to a search strategy. The strategy adopted herein is extremely general, nevertheless it looks suitable for the present purposes. It was previously devised and numerically tested in connection with totally different engineering problems, (Powell, 1964, Nelder-Mead, 1965, Lee-Knapton, 1975).

The starting phase implies the choice (on the basis of engineering judgement) of n+1 parameter vectors, n being the parameter number. These n+1 points define a "simplex" in the n-dimensional space of vectors P (a triangle for n = 2; a tetrahedron for n = 3 ; etc.). The operative sequence of the method is decsribed in the self-explanatory diagram of fig. 7.

An explicit expression is given below for the criterion function ϕ in the elastic model identification. It is assumed as ϕ a norm of the vector of the differences ("error vector") between measured relative displacements v^* and the corresponding theoretical ones. Vector v can be defined by the degree of freedom vector u of the model through a geometric matrix G', coincident with the upper part of G in Eq. (10):

$$v = G' u \qquad (19)$$

Making use of Eqs. (8) (9) and (19), one obtains:

$$\phi = [\underline{v}^* - \underline{G}'(\sum_1^n P_h \underline{K}_h)^{-1} \underline{F}]^T [\underline{v}^* - \underline{G}'(\sum_1^n P_h \underline{K}_h)^{-1} \underline{F}] \quad (20)$$

Clearly, if some components of the load vector $\underline{F}$ are regarded as parameters to identify, the whole procedure can be still applied without significant changes. This extension can be of use in engineering situations exemplified by the case (A) of Sec. 2.

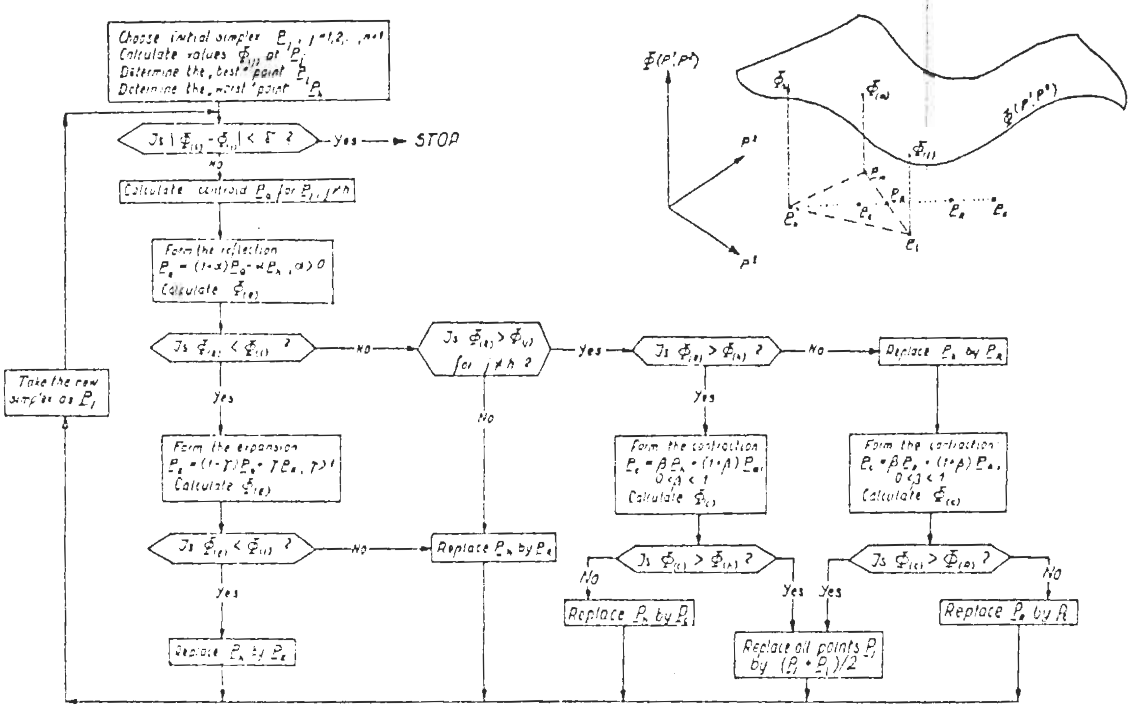

Fig. 7 - Flow chart of a simplex search strategy for identification: $\underline{P}$ parameter vector, ϕ = criterion function, δ = tolerance, α, β, γ = scalars to be chosen.

6. PLASTIC MODEL IDENTIFICATION BY MATHEMATICAL PROGRAMMING WITH COMPLEMENTARITY CONSTRAINTS

When the system to identify is expected to undergo extensive plastic deformations, the main concern of the analysis may be to embody in the model a realistic simulation of the plastic properties. Obviously, these can be determined by identification processes in the sense of this paper, only provided that the experimental data to be used concern situations in which significant parts of the actual systems are plastified. This requirement is sometimes complied with practice.

Under severe restriction on the geometry of the openings, a direct and practical procedure resting on closed form solutions, has been proposed to identify a two-parameter (cohesion and friction angle) yield criterion (Coulomb's) for rock media. (Fanelli-Borsetto, 1976).

In a more general context, recourse could be made to a discrete mathematical model, such that formulated in Sec. 3. Through Eqs. (1), the elastoplastic deformability laws for the i-th finite element can be derived from the material constitutive laws and read:

$$\underline{\psi}_i = \underline{N}_i^T \underline{Q}_i - \underline{k}\, V_i \leq \underline{0} \quad (21)$$

$$\underline{q}_i = \underline{E}_i' \underline{Q}_i + \underline{N}_i \underline{\lambda}_i \;,\; \underline{\lambda}_i \geq \underline{0} \;,\; \underline{\psi}_i^T \underline{\lambda}_i = 0 \quad (22)$$

where:

$$\underline{N}_i = \underline{T}_i^{-1} \underline{n} \;;\; \underline{E}_i' = \underline{T}_i^{-1} \underline{E}^{-1} (\underline{T}_i^{-1})^T V_i^{-1} \quad (23)$$

Let us condense eqs. (21) and (22) for i = 1...m into compact matrix relations concerning the whole (disassembled) model:

$$\underline{\psi} = \underline{N}^T \underline{Q} - \underline{A}\,\underline{k} \leq \underline{0} \quad (24)$$

$$\underline{q} = \underline{E}' \underline{Q} + \underline{N}\,\underline{\lambda} \;,\; \underline{\lambda} \geq \underline{0} \;,\; \underline{\psi}^T \underline{\lambda} = 0 \quad (25)$$

having set:

$$\underline{\psi}^T \equiv (\underline{\psi}_1^T \ldots \underline{\psi}_m^T), \quad \underline{\lambda}^T \equiv (\underline{\lambda}_1^T \ldots \underline{\lambda}_m^T) \quad (26)$$

$$\underline{N} \equiv \text{diag}\,[\underline{N}_i] \;,\; \underline{A} \equiv \text{diag}\,[V_i\, \underline{U}] \;,\; \underline{E}' \equiv \text{diag}\,[\underline{E}_i'] \quad (27)$$

$\underline{U}$ denotes a y-vector with all unit components and $\underline{k} = \underline{P}$ is the vector of the parameters to identify. Combining compatibility and equilibrium Eqs. (2) (3) with (24) (25) one obtains for the displacement vector the expression:

$$\underline{u} = \underline{K}^{-1} \underline{F} + \underline{B}\,\underline{\lambda} \quad (28)$$

where

$$\underline{B} \equiv \underline{K}^{-1} \underline{C}^T \underline{E}' \underline{N} \quad (29)$$

and for the yield function vector $\underline{\psi}$ the expression:

$$\underline{\psi} = \underline{N}^T \underline{Z}\,\underline{N}\,\underline{\lambda} + \underline{B}^T \underline{F} - \underline{A}\,\underline{k} \quad (30)$$

where:

$$\underline{Z} \equiv \underline{E}' \underline{C}\,\underline{K}^{-1} \underline{C}^T \underline{E}' - \underline{E}' \quad (31)$$

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